Buckling of Clamped Oval Cylindrical Shells under Axial Loads

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The effect of clamped boundary conditions on the elastic buckling of finite, oval cylindrical shells under axial compression is investigated. In the development of the problem, a rather accurate representation of the prebuckling state is employed. The rigorous boundary conditions are enforced on both the prebuckling and buckling solutions. The stability equations are solved by the Fourier method in conjunction with the use of a higher-order difference technique. Noticeable deviations between the results of the present analysis and those established for infinitely long shells were observed, particularly for moderate-to-large out-of-roundness.

Nomenclature

f = Airy's stress function
h = uniform wall thickness
p = qr_0/Eh = nondimensionalized uniformly distributed external pressure
q = uniform external pressure
r = local radius of curvature of the cross section of the undeformed middle surface of the shell
r_0 = $L_0/(2\pi)$ = radius of the equivalent circular cylinder
u,v,w = axial, circumferential and inward radial displacements of a point on the middle surface of the shell, respectively
x,s,z = axial, circumferential and radial coordinates, respectively, of any point in the oval shell, and defined according to an orthogonal curvilinear frame of reference
D = $Eh^3/[12(1-\nu^2)]$ = flexural rigidity of the shell
E = Young's modulus of elasticity
F = $\dot{f}/(Ehr_0^2)$ = nondimensionalized first order Airy's stress function
$G^{(l)}$ = prebuckling displacement function
K = $(r_0/h)[12(1-\nu^2)]^{1/2}$ = thickness parameter
L_1,L_0 = axial and circumferential lengths of the cylinder, respectively
M,N = number of terms in series solution for the first order stress function and radial displacement, respectively
M_x,M_s = axial and circumferential bending moment resultants per unit of circumferential and axial length, respectively
M_{xs},M_{sx} = axial and circumferential twisting moment resultants per unit of circumferential and axial length, respectively
$\bar{N}$ = end reaction per unit circumferential length, positive in compression
N_x,N_s = axial and circumferential normal membrane force resultants per unit of circumferential and axial length, respectively
N_{sx},N_{xs} = axial and circumferential membrane shear force resultants per unit of axial and circumferential length, respectively
R = r/r_0 = nondimensionalized local radius of curvature of the oval cross section
S = s/L_0 = nondimensionalized circumferential coordinate
T_i = prebuckling radial displacement function
$U = \dot{u}/L_1$, $V = \dot{v}/L_0$, $W = \dot{w}/r_0$ = nondimensionalized first-order axial, circumferential, and radial displacements of a point on the middle surface of the shell, respectively
$U^0 = u^0/L_1$, $W^0 = w^0/r_0$ = nondimensionalized prebuckling axial and radial displacements of a point on the middle surface of the shell
$X = x/L_1$ = nondimensionalized axial coordinate of any point in the oval shell
α_i = prebuckling displacement parameter
ϵ = unit end shortening
θ = axial stress parameter
θ_{in} = σ_{in}/σ_0 = axial stress parameter at initial buckling
θ_c = σ_c/σ_0 = axial stress parameter at collapse load
ν = Poisson's ratio
ξ = measure of oval eccentricity
σ_0 = $Eh/\{[3(1-\nu^2)]^{1/2}r_0\}$ = buckling stress of an infinite circular cylinder of radius r_0 under axial compression
σ_{in} = average intensity of axial compressive load at initial buckling
σ_c = average intensity of axial compressive load at collapse load
Δ_i = prescribed unit end shortening
$\phi_i(S)$ = correction function for prebuckling axial displacement

1. Introduction

THE use of thin shells as structural elements has been on the increase in recent years due to their high structural efficiency, availability of suitable high-strength materials and advances in the understanding of thin shell behavior. With regard to cylindrical shells, a great many applications, in which stability was a consideration, have been confined to the circular configuration. This, to a large extent, was due to the great deal of work done on the analysis of circular cylindrical shells with the impetus of Donnell's work;[1] see, for example, the extensive bibliographies in Refs. 2–4. Although relatively little theoretical work has been done on the stability of noncircular cylindrical shells, attention can be drawn to works by Marguerre and by Kempner and Chen.[5–9]

In the present study the problem of axial loading of finite length, oval, cylindrical shells with clamped boundaries is investigated. The analysis can be extended to other boundary and loading conditions. The results should be applicable to the design of noncircular cylinders having considerable out-of-

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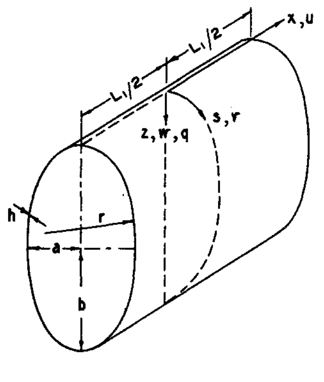

Fig. 1 Sign convention and geometry for coordinates and displacements.

roundness as well as ones designed to be circular, but fabricated slightly oval.

In the development of the problem a representation of the prebuckling state, which proves to be rather accurate within the scope of considerations of static equilibrium, is employed. The corresponding prebuckling deformations appear among the terms of the stability equations and their appearance in turn permits the enforcement of rigorous conditions at the supports of the shell. The dependency of the buckling displacements and the corresponding stresses on the circumferential coordinate are expressed as finite Fourier series, with the Fourier coefficients representing the axial variation. The ordinary differential equations obtained are linear, but possess variable coefficients. They are solved by the finite difference method.

II. Basic Equations

The relations employed in this work are applicable to thin-walled cylindrical shells of uniform thickness, composed of a homogeneous, isotropic, linearly elastic material.

Donnell-type equations[10] are used in the analysis. As is well known, in this type of shell theory, the only equilibrium equation remaining to be satisfied is

$$D\nabla^4 w - f_{,ss}w_{,xx} - f_{,xx}(1/r + w_{,ss}) + 2f_{,xs}w_{,xs} - q = 0 \quad (1)$$

if a stress function f is introduced such that f is related to the membrane stress resultants by

$$N_x = f_{,ss},\ N_s = f_{,xx},\ N_{xs} = -f_{,xs}$$

Moreover, the functions f and w must satisfy the compatibility equation in the form of

$$(1/Eh)\nabla^4 f + w_{,xx}/r - w_{,xs}{}^2 + w_{,xx}w_{,ss} = 0 \quad (2)$$

In addition, the force and moment resultants are related to the middle surface displacements as follows (see Fig. 1):

$$N_x(x,s) = [Eh/(1-\nu^2)]\{u_{,x} + \tfrac{1}{2}w_{,x}{}^2 + \nu[v_{,s} - (w/r) + \tfrac{1}{2}w_{,s}{}^2]\} \quad (3a)$$

$$N_s(x,s) = [Eh/(1-\nu^2)]\{v_{,s} - (w/r) + \tfrac{1}{2}w_{,s}{}^2 + \nu[u_{,x} + \tfrac{1}{2}w_{,x}{}^2]\} \quad (3b)$$

$$N_{xs}(x,s) = N_{sx}(x,s) = [Eh/2(1+\nu)]\{u_{,s} + v_{,x} + w_{,x}w_{,s}\} \quad (3c)$$

$$M_x(x,s) = -[Eh^3/12(1-\nu^2)]\{w_{,xx} + \nu w_{,ss}\} \quad (3d)$$

$$M_s(x,s) = -[Eh^3/12(1-\nu^2)]\{w_{,ss} + \nu w_{,xx}\} \quad (3e)$$

$$M_{xs}(x,s) = -M_{sx}(x,s) = [Eh^3/12(1+\nu)]w_{,xs} \quad (3f)$$

Equations (1) and (2) are two simultaneous partial differential equations which determine the two variables w and f completely, if boundary conditions are properly posed at the ends of the cylindrical shell. In this connection, the natural boundary conditions are such that one member of each of the four products uN_x, vN_{xs}, $w(M_{x,x} - 2M_{xs,s})$ and $w_{,x}M_x$ must be prescribed. Consequently, it is possible to pose four different sets of "clamped" conditions, each of which includes the conditions $w(\pm L_1/2,s) = 0$ and $w_{,x}(\pm L_1/2,s) = 0$ in common. The influence of each of these four sets of boundary conditions on the bifurcation-buckling behavior of circular cylindrical shells (as well as the four variations of simply-supported ends) has been discussed in great detail by Almroth.[11] In the present work, attention is centered on the case "C1" in Almroth's classification, which requires

$$w(\pm L_1/2,s) = w_{,x}(\pm L_1/2,s) = v(\pm L_1/2,s) = 0$$

$$u(\pm L_1/2,s) = \mp\Delta_l/2 = \text{const}$$

at the ends, where Δ_l is the prescribed total end-shortening. The choice is based on the fact that the group of end conditions so prescribed is the only set of "clamped" conditions among the four that can be expected to be satisfactorily reproduced in the laboratory. The analysis that follows, however, can be extended to deal with the other combinations of end conditions without any added difficulties.

Moreover, in order to take full advantage of the expected symmetry of the deformation, it is desirable to consider only one half of the shell. Thus, the boundary conditions posed above are to be enforced at one end, say, $s = L_1/2$ while the stresses and deformations are required to satisfy four conditions of symmetry at the center, viz.,

$$f'(0,s) = 0,\ f'''(0,s) = 0,\ w'(0,s) = 0,\quad w'''(0,s) = 0 \quad (4)$$

An alternative approach to the problem is to solve the three displacement equations of equilibrium, which proves to be advantageous in seeking the prebuckling solution. The displacement equations of equilibrium are obtained by eliminating the stress-resultants in the equilibrium equations by virtue of Eqs. (3). In doing so, it follows that

$$2\{u_{,xx} + w_{,x}w_{,xx} + \nu[v_{,sx} - (1/r)w_{,x} + w_{,s}w_{,sx}]\} + (1-\nu)[u_{,ss} + v_{,xs} + w_{,xs}w_{,s} + w_{,x}w_{,ss}] = 0 \quad (5a)$$

$$2\{\nu[u_{,xs} + w_{,x}w_{,xs}] + v_{,ss} - (1/r)w_{,s} + w_{,s}w_{,ss}\} + (1-\nu)[u_{,sx} + v_{,xx} + w_{,xx}w_{,s} + w_{,x}w_{,sx}] = 0 \quad (5b)$$

$$D\nabla^4 w - [Eh/(1-\nu^2)]\{[(1/r) + w_{,ss}][\nu(u_{,x} + \tfrac{1}{2}w_{,x}{}^2) + w_{,s} - w/r + \tfrac{1}{2}w_{,s}{}^2] + w_{,xx}[u_{,x} + \tfrac{1}{2}w_{,x}{}^2 + \nu(v_{,s} - w/r + \tfrac{1}{2}w_{,s}{}^2)] + w_{,xs}(1-\nu)[u_{,s} + v_{,x} + w_{,x}w_{,s}]\} - q = 0 \quad (5c)$$

III. Prebuckling State

A solution to the nonlinear displacement equations of equilibrium [Eqs. (5)] may be regarded as the combination of the prebuckling deformation (u^0, v^0, w^0) and the buckling deformation ($\dot{u}$, $\dot{v}$, $\dot{w}$) such that $u = u^0 + \lambda\dot{u}$, . . . etc. if the disturbance λ is restricted to be very small. Other quantities, such as the stresses, are also expanded in the same way so that the symbols $(\ )^0$ and $(\dot{\ })$, representing the first two Taylor coefficients (i.e., $(\ )^0 = [(\ )]_{\lambda=0}$, $(\dot{\ }) = [d(\ )/d\lambda]_{\lambda=0}$), can be used as perturbation operators.

For instance, applying the perturbation operator of the zeroth order to Eqs. (5) yields three equations of equilibrium which are identical to the original ones except for the superscripts "0" to be carried by all the displacement components and their derivatives.

From the study of buckling of infinitely long circular cylindrical shells it is clear that the prebuckling displacement field is uniform throughout the entire cylinder. For finite circular cylinders, the prebuckling displacement field is not uniform. It is, however, symmetric with respect to the axis of the cylinder.[11,12] For finite supported oval cylinders, such as the clamped ones in the present work, the prebuckling displacement field is neither uniform nor axisymmetric.[13] Evidently the accurate (if not exact) determination of this nonuniform prebuckling displacements is a prerequisite for the enforcement of the rigorous edge conditions. In this connection a relatively simple approximate solution for the prebuckling state can be obtained within the framework of the following assumptions:[13]

1) The quantities appearing in the zeroth-order version of Eqs. (5) are initially taken to be independent of the circumferential coordinate s; i.e., $\partial(\ )/\partial s = 0$, and r is a constant.

2) The dependency of r upon s is restored for the radius of curvature appearing in the solution of the equations of equilibrium simplified according to the first assumption.

3) The arbitrary function resulting from the partial integration that leads to the determination of the prebuckling axial displacement is permitted to vary in the circumferential direction to insure the uniform movement of the ends; i.e., $u^0(\pm L_1/2,s) = \text{const.}$ These assumptions, hereupon called the "pseudo-symmetry" for easy identification, are justified by the excellent agreement between the simplified solution and the exact solution of the linearized form of Eqs. (5) (Refs. 15 and 16) with the exception of the perimetric displacement v^0. Thus, it seems reasonable to assume that even with the retention of the beam-column terms contained in Eqs. (5), the pseudo-symmetric solution provides an accurate description of the prebuckling state.

The zeroth-order equation of equilibrium derived from the first of Eqs. (5) and simplified utilizing assumption (1) is readily integrable with respect to x, to yield

$$u_{,x}{}^0 + \tfrac{1}{2}w_{,x}{}^{0\,2} - \nu w^0/r = -2(1-\nu^2)\theta\phi/K \qquad (6)$$

in which θ is the load parameter representing the ratio of the average axial stress to the buckling stress $\sigma_0 = 2E/K$ of an infinite circular cylinder of radius r_0; and $K = [12(1-\nu^2)]^{1/2}r_0/h$. This load parameter is related to the average axial load (per unit circumferential length) $\bar{N}$ and the applied external pressure q by

$$\theta = K(\bar{N} + qA/L_0)/2Eh \qquad (7)$$

where A is the area enclosed by the oval cross section and $L_0 = 2\pi r_0$ is the perimetric length of the oval contour. So far, the function $\phi(s)$ is arbitrary except that its average over the oval contour is equal to unity, so that Eq. (6) satisfies over-all equilibrium in the axial direction.

Finally, when assumption (1) is applied to the last of Eqs. (5) and when the equation obtained is combined with Eq. (6) to eliminate $u_{,x}{}^0$, the following equation governing the nondimensional deflection W^0 results

$$W^0{}_{,XXXX} + 2K(L_1/r_0)^2\theta\phi W^0{}_{,XX} + (K/R)^2(L_1/r_0)^4W^0 = (K/R)^2(L_1/r_0)^4(pR^2 - 2\nu R\theta\phi/K) \qquad (8)$$

in which the following definitions have been introduced

$$X = x/L_1,\ W^0 = w^0/r_0,\ R(S) = r/r_0,$$
$$p = qr_0/Eh,\ S = s/L_0$$

Equation (8), with the exception of the second term (the "beam-column term"), was obtained and solved in Ref. 13. It is exact for the circular cylinder to the extent of Donnell's (or Timoshenko's) accuracy when ϕ and R are both equal to unity (see, for instance, Refs. 11 and 12). It can be easily shown that the solution to Eq. (8) appears formally as:

$$W^0 = (pR^2 - 2\nu R\theta\phi/K)T_i \qquad (i = 1,2,3) \qquad (9)$$

which is the same form as that obtained in Ref. 13. However, with the beam-column term retained, the function T_i takes on various forms. These relations depend upon whether the quantity $R\theta\phi$ (proportional to the magnitude of the applied load) is less than, greater than, or equal to unity, corresponding to values of i of 1, 2 or 3, respectively. Thus,

$$T_1 = 1 - C_1\sin 2\alpha_1X\sinh 2\alpha_2X - C_2\cos 2\alpha_1X\cosh 2\alpha_2X$$
$$T_2 = 1 - C_3\cos 2\alpha_3X - C_4\cos 2\alpha_4X$$
$$T_3 = 1 - C_5X\sin 2\alpha_5X - C_6\cos 2\alpha_5X$$

where

$$\alpha_1,\alpha_2 = \tfrac{1}{2}(L_1/r_0)[K(1 \pm R\theta\phi)/2R]^{1/2} \qquad (10a)$$

$$\alpha_3,\alpha_4 = \tfrac{1}{2}(L_1/r_0)\{[K(R\theta\phi + 1)/2R]^{1/2} \pm [K(R\theta\phi - 1)/2R]^{1/2}\} \qquad (10b)$$

$$\alpha_5 = \tfrac{1}{2}(L_1/r_0)[K/R]^{1/2} \qquad (10c)$$

The constants of integration C_i are evaluated by means of the clamped-end conditions

$$W^0(\pm\tfrac{1}{2},S) = 0 \text{ and } W^0{}_{,X}(\pm\tfrac{1}{2},S) = 0$$

The nondimensional axial displacement $U^0 = u^0/L_1$ can now be obtained by integrating Eq. (6) with the condition of symmetry $U^0(0,S) = 0$. The results of the straightforward integration of the linearized form of Eq. (6) for the three cases will not be presented here. All three cases are subjected to the condition that the end-shortening must be uniform, i.e., $U^0(\tfrac{1}{2},S) = -\epsilon/2$ where the constant unit end-shortening ϵ is defined by $\epsilon = \Delta_l/L_1$, with Δ_l being the total shortening of the cylinder. When this condition is enforced, it follows that

$$\epsilon_i = (\rho\nu R - 2\theta\phi/K) - (p\nu R - 2\nu^2\theta\phi/K)\psi_i$$

in which $i = 1, 2$ and 3 correspond to the three cases for which $R\theta\phi$ is less than, greater than, or equal to unity, and the functions ψ_i corresponding to these cases are:

$$\psi_1 = [2\alpha_1\alpha_2/(\alpha_1^2 + \alpha_2)^2][(\cosh 2\alpha_2 - \cos 2\alpha_1)/(\alpha_1\sinh 2\alpha_2 + \alpha_2\sin 2\alpha_1)]$$

$$\psi_2 = \left(\frac{\alpha_3^2 - \alpha_4^2}{\alpha_3\alpha_4}\right)\left[\frac{\sin\alpha_3\sin\alpha_4}{\alpha_3\sin\alpha_3\cos\alpha_4 - \alpha_4\sin\alpha_4\cos\alpha_3}\right]$$

$$\psi_3 = (2/\alpha_5)[\sin^2\alpha_5/(\alpha_5 + \sin\alpha_5\cos\alpha_5)]$$

According to the second pseudosymmetric assumption, the quantities R and ϕ are now regarded as functions of S. Consequently, α_j and ψ_i are functions of S while T_i, $W_i{}^0$ and $U_i{}^0$ are functions of X and S.

One final preparation to facilitate the determination of the function ϕ is to rewrite the expression for the unit end-shortening as:

$$\epsilon_i = -(\phi/F_{1i})(2\theta/K - \nu pF_{2i}/\phi)$$

where

$$F_{1i} = F_{1i}(S) = 1/(1 - \nu^2\psi_i),$$
$$F_{2i} = F_{2i}(S) = R(1 - \psi_i)/(1 - \nu^2\psi_i)$$

Let the symbol $(\tilde{\ })$ be introduced to indicate an average value of $(\ )$; i.e.,

$$(\tilde{\ }) = \frac{1}{L_0}\int_0^{L_0}(\ )ds = \int_0^1(\ )dS$$

Then, as stated earlier, $\tilde{\phi} = 1$, and, moreover, since the unit end-shortening is independent of S, it follows that $\epsilon_i = \tilde{\epsilon}_i$. When these two conditions are enforced,

$$\phi = \phi_i(S) = (F_{1i}/\tilde{F}_{1i}) - (\nu Kp\tilde{F}_{2i}/2\theta)(F_{1i}/\tilde{F}_{1i} - F_{2i}/\tilde{F}_{2i})$$

Finally, upon the elimination of ϕ from Eq. (9), it follows that:

$$W_i{}^0 = pRT_i[R + \nu^2\tilde{F}_{2i}F_{1i}/\tilde{F}_{1i} - \nu^2F_{2i}] - 2\nu R\theta T_iF_{1i}/K\tilde{F}_{1i}$$

It should be noted, however, that up to this point $\phi(S)$ remains indeterminate. The complete determination of $\phi(S)$

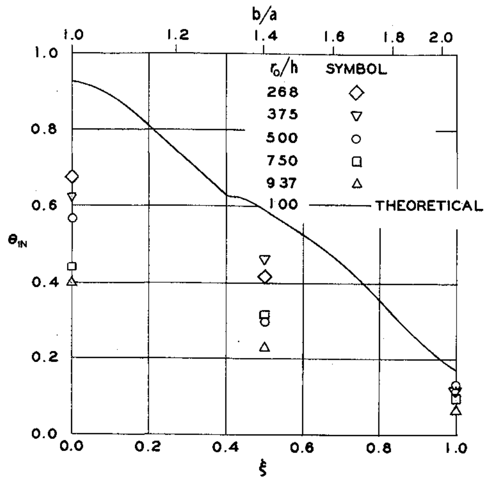

Fig. 2 Initial buckling stress.

for a given value of θ relies upon a numerical iterative process discussed in detail in a later section.

IV. Stability Equations

The stability equations can be obtained by the application of the first-order perturbation operator $(\dot{\ })$ to Eqs. (5). However, as a matter of convenience, this operation is performed, instead, on Eqs. (1) and (2). The quantities appearing in the resulting stability equations are w^0, f^0, $\dot{w}$, $\dot{f}$ and their derivatives. In these equations the prebuckling force-resultants expressed in the form of $f^0{}_{,xx}, \ldots$, etc. are calculated from the zeroth-order form of Eqs. (3) which yield

$$N_x{}^0 = f^0{}_{,ss} = -2Eh\theta\phi/K \qquad (11a)$$

$$N_s{}^0 = f^0{}_{,xx} = -Eh[W^0/R + 2\nu\theta\phi/K] \qquad (11b)$$

$$N_{xs}{}^0 = -f^0{}_{,xs} = 0 \qquad (11c)$$

Of course, the stability equations so obtained would be equivalent to their counterparts derivable as first order versions of Eqs. (5) if w^0 and the resultant forces in Eqs. (11) were exact. Nevertheless, the above equivalence holds within the accuracy of the pseudo-symmetric approximation, since the functions w^0 and $f^0{}_{,xx}$, $f^0{}_{,ss}$, $f^0{}_{,xs}$ given in Eqs. (11) do satisfy the zero-order version of Eqs. (1) and (2) to the extent of the assumptions of pseudo-symmetry.

After the zeroth-order stress function f^0 is eliminated by virtue of Eqs. (11), the stability equations are obtained with the aid of the aforementioned procedure and cast into the following dimensionless form:

$$W_{,XXXX} + 2(L_1/L_0)^2W_{,XXSS} + (L_1/L_0)^4W_{,SSSS} + 8\pi^2K\theta\phi(L_1/L_0)^2W_{,XX} + 4\pi^2(L_1/L_0)^4[2\nu\theta\phi K + (K^2/R)W^0]W_{,SS} - K^2(L_1/L_0)^2W^0{}_{,XX}F_{,SS} - 4\pi^2K^2(L_1/L_0)^2F_{,XX}/R = 0 \qquad (12)$$

$$F_{,XXXX} + 2(L_1/L_0)^2F_{,XXSS} + (L_1/L_0)^4F_{,SSSS} + 4\pi^2(L_1/L_0)^2W_{,XX}/R + (L_1/L_0)^2W^0{}_{,XX}W_{,SS} = 0 \qquad (13)$$

in which the nondimensionalized variables W and F are defined as

$$W = W(X,S) = \dot{w}/r_0,\ F = F(X,S) = \dot{f}/(Ehr_0{}^2)$$

Although extremely complicated, the variable coefficients appearing in Eqs. (12) and (13) are known functions that have been obtained in Section III. Since these coefficients are derivable from the determined function $W^0(X,S)$, however complicated, a systematic representation is made possible by the following Fourier synthesis of W^0:

$$W_i{}^0 = \tfrac{1}{2}G_i{}^{(0)}(X) + G_i{}^{(1)}(X)\cos 4\pi S + G_i{}^{(2)}(X)\cos 8\pi S \qquad (14)$$

in which the Fourier coefficients are given by

$$G_i{}^{(m)}(X) = 8p\int_0^{\frac{1}{4}} R\left[R + \nu^2\tilde{F}_{2i}\left(\frac{F_{1i}}{\tilde{F}_{1i}}\right) - \nu^2F_{2i}\right] \times T_i\cos 4m\pi S dS - \frac{16\nu\theta}{K}\int_0^{\frac{1}{4}} RT_i\frac{F_{1i}}{\tilde{F}_{1i}}\cos 4m\pi S dS$$

In the above relations, the subscript i is equal to 1, 2 or 3 depending whether the quantity $R\theta\phi$ is less than, greater than, or equal to unity. The three-term Fourier series given in Eq. (14) was suggested in Refs. 13 and 14, since such an approximation gives extremely good agreement with the exact linearized solution of Eqs. (5). It is, therefore, assumed here that the three-term representation also closely approximates the solution of Eqs. (5) with the beam-column term retained.

In the present work, the local radius of curvature R is represented by:

$$R = 1/(1 + \xi\cos 4\pi S) \qquad (15)$$

where $0 \leq \xi \leq 1$ is a measure of the eccentricity of the cross section of the shell (Fig. 1). Such a relation describes a doubly-symmetric oval cross section. This expression for the nondimensional local radius of curvature was chosen because it produces a family of closed curves of constant circumferential length with an easily definable out-of-roundness,[15] which provides a convenient basis for comparison of the stability of noncircular cylinders to that of a circular one.

The nondimensionalized stress function F and buckling deflection W may be assumed to be

$$F(X,S) = \tfrac{1}{2}A^{(0)}(X) + \sum_{m=1}^{M} A^{(m)}(X)\cos 4m\pi S \qquad (16a)$$

$$W(X,S) = \tfrac{1}{2}B^{(0)}(X) + \sum_{n=1}^{N} B^{(n)}(X)\cos 4n\pi S \qquad (16b)$$

Observe that these relations are comparable to those used by Almroth[11] with the exception of the difference in the circumferential wave number. These relations imply the assumption of double symmetry in the buckle pattern, while Almroth only assumed symmetry with respect to one axis. Such an assumption of symmetry was motivated by the result of the analysis of the infinitely long oval shells[9] that odd and even harmonics do not interact. Moreover, with the possible exception of supported cylinders of extremely small length, these harmonics couple only weakly, as may be seen from Ref. 11.

The two groups of functions $A^{(m)}(X)$ and $B^{(n)}(X)$ in Eqs. (16) are governed by a set of coupled ordinary differential equations with known variable coefficients. These equations can be obtained following the substitution of Eqs. (16) into Eqs. (12) and (13), and thereafter finite-difference arithmatic takes over.

V. Boundary Conditions

The boundary conditions on the shell must be represented as relations involving the functions $A^{(m)}$ and $B^{(n)}$.

Table 1[a] Convergence of series test for initial buckling stress parameter θ_{in}

ξ	0.1	0.3	0.5	0.7	0.9
M = N = 4	0.884	0.704	0.536	0.424	0.239
M = N = 6	0.876	0.702	0.533	0.424	0.239

[a] $r_0/L_1 = 0.200$; $r_0/h = 100$; $P = 10$.

The clamped-end boundary conditions to be satisfied at $x = L_1/2$ during buckling are:

$$\dot{u}(L_1/2, s) = 0, \dot{v}(L_1/2, s) = 0 \quad (17a)$$

$$\dot{w}(L_1/2, s) = 0, \dot{w}'(L_1/2, s) = 0 \quad (17b)$$

while the conditions of symmetry are prescribed at the midsection of the shell by Eqs. (4). It should be noted that the assumption of antisymmetry instead of symmetry or even the combination of both is equally plausible. Although analyses based upon such conditions are not carried out, it is believed that the results that follow would not differ substantially from what are actually obtained.

Equations (16) were substituted directly in the nondimensional form of Eqs. (4) and (17b) to obtain six simple equivalent relations among the functions $A^{(m)}$ and $B^{(n)}$.

Additional boundary conditions on the function $A^{(m)}$ are obtained from Eqs. (17a) by utilizing Eqs. (3) and (16) to obtain

$$A^{(m)''}(\tfrac{1}{2}) + 4\nu m^2(L_1/r_0)^2 A^{(m)}(\tfrac{1}{2}) = 0 \qquad m = 0, 1, \ldots, M$$

and

$$A^{(m)'''}(\tfrac{1}{2}) - 4m^2(2 + \nu)(L_1/r_0)^2 A^{(m)'}(\tfrac{1}{2}) = 0 \qquad m = 0, 1, \ldots, M$$

corresponding to $\dot{v}(L_1/2,s) = 0$ and $\dot{u}(L_1/2,s) = 0$.

VI. Solution of the Stability Equations

As far as can be ascertained, there is no feasible way to obtain closed form solutions to the boundary value problem posed in the preceding sections. The solution is therefore developed utilizing the finite difference method, modified to handle the present problem. The general procedures of the finite difference technique will not be elaborated here; but the following aspects should be noted:

1) In the preliminary calculations, the arbitrary function $\phi(S)$ and the dimensionless wave number $\alpha(S)$ must be determined. The fact that $\phi(S)$ depends upon α is obvious in the development of the prebuckling solution. On the other hand, the determination of the subscript i in $\alpha = \alpha_i$ and thus the choice of an appropriate relation among Eqs. (10) depends upon the local values of the product $\phi\theta R$. This dictates the use of an iterative loop starting arbitrarily with, say, $\phi = 1$ for all values of S.

2) During the transition from the partial differential equations [Eqs. (12) and (13)] to the ordinary differential equations which ultimately led to the difference equations, the original equations were multiplied by $\cos j\pi S dS$ and a series of numerical integrations with respect to the variable S were performed by means of the formulas of Weddle[16] and Filon.[17]

Table 2[a] Initial buckling stress parameter θ_{in} for various numbers of axial subdivisions

ξ/P	10	12	14	16	18	20
0.0	0.951	0.940	0.884	0.915	0.942	0.925
0.1	0.884	0.887	0.859	0.885	0.897	0.889
0.2	0.799	0.800	0.801	0.809	0.814	0.814
0.3	0.704	0.704	0.708	0.719	0.719	0.719
0.4	0.617	0.617	0.630	0.630	0.630	0.630
0.42	...	...	...	...	...	0.625
0.45	...	...	...	...	...	0.619
0.5	0.536	0.536	0.569	0.569	0.579	0.587
0.6	0.484	0.484	0.509	0.523	0.523	0.523
0.7	0.424	0.425	0.456	0.456	0.456	0.456
0.8	0.347	0.356	0.358	0.358	0.358	0.358
0.9	0.239	0.255	0.255	0.255	0.255	0.255
1.0	0.146	0.146	0.146	0.146	0.171	0.171

[a] $r_0/L_1 = 0.200$; $r_0/h = 100$.

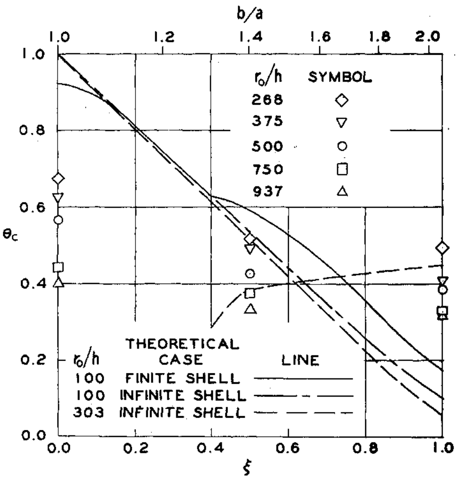

Fig. 3 Collapse stresses.

3) If conventional central difference operators were employed, the secular determinant formed of the coefficients of the quantities $A_i^{(m)} = A^{(m)}(X_i)$, $B_i^{(n)} = B^{(n)}(X_i)$ (X_i being a representative pivotal point) would be large in size and narrowly banded. To make a more efficient use of the unoccupied cells of the banded matrix, higher-order difference operators should be used. The basic difficulties such higher-order operators present are focused in the boundary conditions, in that a larger number of such conditions is needed but is not available. This was pointed out by Greenwood, and counter measures such as imposing some artificial conditions of symmetry and antisymmetry at the boundary pivot were proposed.[18] As the advantages (in speed of convergence in terms of numbers of pivots required) are weighed against the added difficulties, it was found that the operators most suitable for the present problem are the ones associated with errors of the order Δ^6, in contrast to the usual Δ^2 error for the ordinary central difference operators, were Δ is the length of an interval.

VII. Discussion and Conclusions

Initial (classical) buckling loads were computed using the numerical methods described in the previous section. The computations were performed for the family of shells described by Eq. (15), having the following properties:

$$\nu = 0.3, r_0/L_1 = 0.2, r_0/h = 100$$

and under a central axial load with no applied lateral pressure. The results are listed in Tables 1 and 2 and plotted in Figs. 2 and 3.

From the results shown in Table 1, it is seen that the determination of the initial buckling load θ_{in} is satisfactory with four term series solutions for the stress function F and the displacement W.

In Table 2 are listed the values of the axial stress parameter θ_{in} for initial buckling of shells of various geometries ($\xi = 0.0$ to $\xi = 1.0$), with the shells divided into increasing numbers of axial segments ($P = 10$ to $P = 20$ for one half the shell length). From this it is seen that there is a rapid convergence of results, indicating that a relatively small number of segments is required to attain an acceptable accuracy, with the possible exception of the case $\xi = 0.5$. This anomaly seems to be due to the critical position of this geometry (see Fig. 3). For clamped circular and nearly circular shells ($0.0 \leq \xi \lesssim 0.16$) there is a small reduction in the load bearing capacity prior to buckling, as compared to the infinite shell, while in the range $0.16 \lesssim \xi \lesssim 0.4$ the solutions are virtually identical to those for an infinitely long shell.[9] Above $\xi \approx 0.4$ the curve for the clamped shell branches up above that for the infinite shell exhibiting a large increase in the buckling load for the clamped end condition.

The deviation of the present results from those for the corresponding infinite shell as described above and shown in Fig. 3 may be explained in several ways. For example, the boundary conditions produce two effects. In the first they cause the prebuckling stress distribution of the finite shell to be altered from that of the infinite shell (with its uniform stress distribution) due to the difference in their prebuckling deformations, while in the second they restrict the prebuckling deformation at the ends of the shell; i.e., they tend to stiffen the shell. The combination of these effects may produce either an increase of a decrease in the initial buckling load, depending on the geometry of the shell in question. For instance, due to the rotational symmetry of the circular shell, all points on the shell are equally vulnerable to buckling, and any nonuniformity in stress due to the first effect will produce regions of higher stress which will cause the circular shell to fail at a reduced load, while the second effect has little influence on this shell. However, in the case of highly noncircular shells, in the regions of the shell with a small curvature, where the initial buckling occurs, the first effect may redistribute the load so as to reduce the stress in these regions, while the second effect, in addition, increases its resistance to buckling due to the stiffening effects of the edge conditions.

In addition, the effects of the prebuckling deformation that tend to weaken the circular shell ($\xi = 0$) fade as ξ increases toward unity. This aspect can best be demonstrated by considering the extreme case of $\xi = 1$. In the neighborhood surrounding an end of the minor axis where buckles initiate, the local value of the product $R\theta\phi$ for such a shell must be greater than unity since both ϕ and θ are finite which R is infinite. Thus, the sinusoidal mode T_2 prevails instead of the rapidly decaying mode T_1 of which the boundary layer effects contribute heavily toward the weakening of the cylinder.

Some of the curves drawn in Fig. 3 are based on theoretical results obtained by Kempner and Chen in Refs. 8 and 9. In Fig. 3 the dashed curves going from the upper left to the lower right are for the initial buckling load for various shell thicknesses ($r_0/h = 100$ and 303). While the curve running approximately horizontally across the right-hand side, is a plot of the maximum load carried by the shell ($r_0/h = 303$) after buckling has taken place.

The individual points plotted in Figs. 2 and 3 are experimental results and were obtained from Ref. 19. The points in Fig. 2 represent the maximum values of initial buckling stresses obtained from tests on clamped shells of the given geometries, while those in Fig. 3 are the maximum experimental collapse stresses for the same group of cylinders.

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